GEOMETRY AND COMBINATORICS OF

STRONG MEASURE ZERO

IN METRIC SPACES AND POLISH GROUPS

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FIRST THINGS FIRST

- Strong Measure Zero
- Borel Conjecture
- Examples
- Hausdorff Measures
- Products
- The Game
- Partition Arrow

DET (BOREL 1919)

A set of reals $X \subseteq \mathbb{R}$ has **STRONG MEASURE ZERO** if for any sequence $\langle \varepsilon_n : n\in\omega\rangle$ of pos. numbers there is a cover $\{E_n : n\in\omega\}$ such that

$$\forall n \quad \operatorname{diam} E_n \le \varepsilon_n.$$

"$\langle\varepsilon_n\rangle$-fine cover"

→ generalizes to metric spaces

→ equivalently if $\exists$ an $\langle\varepsilon_n\rangle$-fine λ-cover

$$\forall x\in X \ \exists^\infty n \quad x\in E_n$$

- $\omega = \bigcup_k I_k$ partition
- $\forall k$ .. $\{E_n : n\in I_k\}$ $\langle\varepsilon_n : n\in I_k\rangle$-fine cover
- $\{E_n : n\in\omega\} = \bigcup_{k\in\omega} \{E_n : n\in I_k\}$

BOREL CONJECTURE

CONJECTURE (Borel 1919)

Every S_{mz} set of reals is countable.

- $\neg BC$ is consistent

→ Sierpinski 1928: A Luzin set is S_{mz}

- Luzin set: $X \subseteq \mathbb{R}$ $\forall M$ meager $|X \cap M| \leq \omega$
- $\langle \varepsilon_n \rangle$ given
- $\mathbb{Q} = \{q_n\}$ rationals: $G = \bigcup_n B(q_n, \varepsilon_{2n})$ dense open
- hence $X \setminus G = \{x_n\}$ ... $X \setminus G \subseteq \bigcup_n B(x_n, \varepsilon_{2n+1})$

→ CH => $\exists$ Luzin set

→ GÖDEL 1938, 1940: consistency of CH

BOREL CONJECTURE

- Other examples
  - → $(\mathrm{cov}\,\mathcal{M} = \mathfrak{c})$ X Sm$_z$, $|X| = \mathfrak{c}$
  - → $(\mathfrak{b} = \omega_1)$ X Sm$_z$, $|X| = \omega_1$ Rothberger 1941
- LAVER 1976: BC is consistent
- Carlson 1993:
  - → If X is a separable metric space, $|X| < \mathfrak{c}$ then $X \xrightarrow[\text{dep.}]{} \mathbb{R}$
  - → BC $\Rightarrow$ no uncountable Sm$_z$ separable metric space
    - If X is Sm$_z$, $\omega < |X| < \mathfrak{c}$ then $f(X) \subseteq \mathbb{R}$ is Sm$_z$

ELEMENTARY PROPERTIES

- All spaces are separable metric spaces
- $Sm_Z(X)$ ... all Sm_Z subsets of a space X

→ $Sm_Z(X)$ is a σ-ideal

→ Sm_Z is preserved by uniformly continuous maps

X Sm_Z, $f: X \to Y$ uif $\Rightarrow$ $f(X)$ Sm_Z

→ Sm_Z is not a metric property

→ Sm_Z is not a topological property

Miller, Fremlin 1988 : TFAE

(i) X is Sm_Z in each compatible metric

(ii) X has the Rothberger property

for each seq. $\langle \mathcal{U}_n \rangle$ of open covers there is a "diagonal" cover $\{U_n\}$, $U_n \in \mathcal{U}_n$

→ If X is σ-compact then $Sm_Z(X)$ is independent of the metric

HAUSDORFF MEASURES

→ gauge (or Hausdorff function): $g:[0,\infty)\to[0,\infty)$

- g nondecreasing
- right-continuous
- $g(0)=0$, $g(r)>0$ for $r>0$

→ g-dimensional Hausdorff measure:

$$\mathcal{H}^g_\delta(E) = \inf\left\{\sum_n g(\operatorname{diam} E_n) : \operatorname{diam} E_n \le \delta,\ \{E_n\} \text{ covers } E\right\}$$

$$\mathcal{H}^g(E) = \lim_{\delta\to 0} \mathcal{H}^g_\delta(E)$$

→ $\mathcal{H}^g(E)=0 \iff \exists\ \lambda\text{-cover } \{E_n\} \text{ — } \sum g(dE_n)<\infty$

$\iff \forall \varepsilon>0\ \exists$ cover $\{E_n\}$

$\cdots \sum g(dE_n)<\varepsilon$

Sm_2 versus HAUSDORFF MEASURES

THM (Besicovitch 1933) TFAE

(i) X is Sm_2

(ii) For each gauge g ... $\mathcal{H}^g(X) = \infty$

Proof (i) $\Rightarrow$ (ii)

- g gauge, $\delta > 0$
- $\langle \varepsilon_n \rangle$ such that $\forall n \; g(\varepsilon_n) < 2^{-n-1}\delta$
- $\{E_n\}$ $\langle \varepsilon_n \rangle$- fine cover of X
- $\sum g(dE_n) \leq \sum g(\varepsilon_n) \leq \sum 2^{-n-1}\delta = \delta$

(ii) $\Rightarrow$ (i)

- $\langle \varepsilon_n \rangle$ arbitrary
- g gauge such that $g(\varepsilon_n) > \frac{1}{n}$
- $\{E_n\}$ s.t. $\sum g(dE_n) < 1$
- $\langle g(dE_n) \rangle$ decreasing ....

.... $g(dE_n) < \frac{1}{n} < g(\varepsilon_n) \Rightarrow dE_n < \varepsilon_n$

SMZ versus HAUSDORFF MEASURES

THM TFAE

(i) X is SMZ

(ii) $\mathcal{H}^g(X) = 0$ for each gauge g

(iii) $\dim_H(X,\rho) = 0$ for each uniformly equivalent ρ.

(iv) $\dim_H f(X) = 0$ for each uniformly cont. map f.

⟶ If X is SMZ then

- $\dim_H X = 0$
- $\dim X = 0$ (Kuratowski 1968)

⟶ SMZ ~ "universal Hausdorff measure zero"

UNIVERSAL MEASURE ZERO

DEF X has **universal measure zero** if every finite Borel, non-atomic measure on X is trivial. UMZ

→ Szpilraju 1933: Every SmZ set is UmZ

Proof
- Suppose μ is a Borel measure on X, no atoms, $0 < \mu X < \infty$.
- $\phi(r) := \sup\{\mu E : \operatorname{diam} E \leq r\}$
  - ↳ ϕ is a gauge
- $\mathcal{H}^{\phi}(X) \geq \mu(X) > 0$

→ Oz 2012: There is UmZ set $X \subseteq \mathbb{R}$, $\dim_H X = 1$

→ Oz 2003: (non CH) $\exists$ UmZ $X \subseteq \mathbb{R}^2$, $\dim X = 1$

PRODUCTS

→ Closed measure zero sets (in 2^ω or $\mathbb{R}$)

$\mathcal{E}$ -- the σ-ideal generated by compact sets of Lebesgue measure zero

→ For any gauge there is a cpt set $C \subseteq \mathbb{R}$

- $\mathcal{H}^g(C) = 0$ (in part $\dim_H C = 0$)
- $\mathcal{H}^1(C \times C) > 0$

THM (Oz 2019) Let X be a metric space. TFAE

(i) X is Sm

(ii) $\mathcal{H}^g(X \times Y) = 0$ whenever Y is cpt and $\mathcal{H}^g(Y) = 0$

(iii) $\mathcal{H}^1(X \times E) = 0$ for all $E \in \mathcal{E}$

PRODUCTS

Proof of (i) => (ii)

- X Smz, g gauge, Y cpt, $\mathcal{H}^g(Y) = 0$
- Let $\delta > 0$.
- $\forall j$ $\exists$ finite cover $\mathcal{U}_j$ of Y... $\sum_{U \in \mathcal{U}_j} g(dU) < 2^{-j-1}\delta$
- $\varepsilon_j := \min_{U \in \mathcal{U}_j} dU$
- $\exists$ $\langle \varepsilon_j \rangle$-fin cover $\{V_j\}$ of X
- $\mathcal{W} = \{ \underbrace{V_j \times U}_{\text{diam } V_j \times U = \text{diam } U} : U \in \mathcal{U}_j \}$ covers $X \times Y$
- $\sum_{W \in \mathcal{W}} g(dW) = \sum_j \sum_{U \in \mathcal{U}_j} g\,dU \leq \delta$

$\Longrightarrow \mathcal{H}^g(X \times Y) = 0$

PRODUCTS

THM (Pawlikowski 1993)

$X \subseteq 2^\omega$ is Smz $\iff$ $X + E$ has measure zero for each $E \in \mathcal{E}$

→ "$\Rightarrow$" $\mathcal{H}^1(X \times E) = 0$ ⋯ $X + E = \text{proj}(X \times E)$

↳ Lipschitz

EXAMPLE ($\text{cov}\,\mathcal{M} = \mathfrak{c}$) $\exists\, X \subseteq 2^\omega$

- X is Smz
- $X + X = 2^\omega$
- $X \times X$ is <u>not</u> Smz

→ if $X \times X$ were Smz, $X + X$ would be Smz.

GALVIN'S GAME

→ GAME: Player I ε_0 ε_1 ε_2 ...

Player II E_0 E_1 E_2 ...

→ RULES
- $\varepsilon_n > 0$
- $\operatorname{diam} E_n \leq \varepsilon_n$

→ II wins if $\{E_n\}$ covers X

THM (Galvin 2017) Let X be T_1-totally bounded. TFAE

(i) X is σ-compact

(ii) I does not have a winning strategy

NOTE (Galvin 2017) $|X| \leq \omega \iff$ II has a winning strategy

GALVIN'S GAME

Proof of $\mathrm{Sm}_2 \Rightarrow$ I has no winning strategy

→ Suppose • X totally bounded

• σ ... strategy

→ Recursion: $\delta_n > 0$, $\mathcal{B}_n$ finite families

• $\delta_n = \min\{\sigma(B_0, B_1, \dots, B_{n-1}) : B_i \in \mathcal{B}_i\}$

• $\mathcal{B}_n$ ... finite family of sets:

→ $\forall B \in \mathcal{B}_n \quad dB \le \delta_n$

→ $dE \le \frac{\delta_n}{3} \Rightarrow \exists B \in \mathcal{B}_n \quad E \subseteq B$

→ $\{E_n\}$ .. $\langle \frac{\delta_n}{3} \rangle$- fine cover of X

→ $\exists B_n \in \mathcal{B}_n \quad E_n \subseteq B_n$ ... $\{B_n\}$ $\langle \delta_n \rangle$-fin cover

→ $\varepsilon_n := \sigma(B_0, B_1, \dots, B_{n-1})$

→ ε_0 B_0 ε_1 B_1 .... o played according to σ

o get II wins

PARTITION ARROW

→ open cover

→ open n-cover : $\forall F \in [X]^n \; \exists U \in \mathcal{U} \quad F \subseteq U$

→ open ω-cover : $\forall F \in [X]^{<\omega} \; \exists U \in \mathcal{U} \quad F \subseteq U$

$\equiv$ n-cover for each $n \in \omega$

→ uniform cover: $\exists \varepsilon > 0 \; \forall x \; \exists U \in \mathcal{U} \quad B(x, \varepsilon) \subseteq U$

→ uniform n-cover: $\exists \varepsilon > 0 \; \forall F \in [X]^n \; \exists U \in \mathcal{U}$

$B(F, \varepsilon) \subseteq U$

$\hookrightarrow \bigcup_{x \in F} B(x, \varepsilon)$

→ uniform ω-cover : uniform n-cover for each n

$\equiv \exists \langle \varepsilon_n \rangle \quad \forall F \in [X]^{<\omega} \; \exists U \in \mathcal{U} \quad B(F, \varepsilon_{|F|}) \subseteq U.$

→ Ω^{unif} ... class of all uniform ω-covers

PARTITION ARROW

DEF $\mathcal{A}$, $\mathcal{B}$ classes of covers

$$\mathcal{A} \longrightarrow (\mathcal{B})^n_k$$

$\forall\, \mathcal{U} \in \mathcal{A}\ \ \forall\, \chi : [\mathcal{U}]^n \to k$

$\exists\, \mathcal{V} \in \mathcal{B}$ χ-homogeneous, $\mathcal{V} \subseteq \mathcal{U}$

THM (Oz 2022+, based on Scheepers 1999)

Let X be T_1-totally bounded. TFAE

(i) X is Sm_2 $\qquad$ $\mathcal{O}$... open covers

(ii) $\Omega^{unif} \longrightarrow (\mathcal{O})^2_2$

(iii) $\Omega^{unif} \longrightarrow (\mathcal{O})^2_k$ for all k

GALVIN-MYCIELSKI-SOLOVAY THEOREM

AND ITS PITFALLS

- Prikry and GMS Theorems
- Extension to locally compact groups
- Failure for non-locally compact groups

THM (Galvin, Mycielski, Solovay 1976)

A set $X \subseteq \mathbb{R}$ is Smz if and only if

$X + M \neq \mathbb{R}$ for each meager set $M \subseteq \mathbb{R}$.

↳ $= \{x+m : x \in X, m \in M\}$

POLISH GROUPS

- Polish group : topological group
  separable, completely metrizable

- Every Polish group
  - → has a complete metric
  - → has a left-invariant metric
  - → but some may have no metric that is both complete and left-invariant
  - → every (two-sided) invariant metric is complete
  - → CLI group : has a complete left-invariant metric
  - → TSI group : has an invariant
    - → every abelian or compact is TSI

SMZ IN POLISH GROUPS

DEF Let $\mathbb{G}$ be a Polish group

$X \subseteq \mathbb{G}$ has (left) strong measure zero if

→ it is SMZ w.r.t. (any) left-invariant metric

→ equivalently, if for any sequence $\langle U_n \rangle$ of open sets (equiv. nbhds of 1) there is a sequence $\langle x_n \rangle$ in $\mathbb{G}$ s.t. $X \subseteq \bigcup_n x_n U_n$

→ $SMZ(\mathbb{G})$ is bi-invariant

→ but if X is (left) SMZ, X^{-1} does not have to:

Hrušák, Zapletal 2016: (cov $\mathcal{M}$ = $\mathfrak{c}$) $\exists X \subseteq S_\infty$ s.t. X SMZ, X^{-1} not SMZ

→ but if $\mathbb{G}$ is σ-compact, then X SMZ $\Leftrightarrow$ X^{-1} SMZ

PRIKRY'S THEOREM

THM (Prikry 1970) Let $X \subseteq G$.
If $XM \neq G$ for each M meager, then X is Smz.

Proof

- Let $Q \subseteq G$ be countable dense, g gauge
- $\exists\, U \supseteq Q$ G_δ-set such that $\mathcal{H}^g(U) = 0$
- $M := G \setminus U^{-1}$ is meager
- $\exists\, z \notin XM$ .... $X \subseteq zU$
- $\mathcal{H}^g(X) \leq \mathcal{H}^g(zU) = \mathcal{H}^g(U) = 0$
  ↳ left invariance

CONVERSE TO PRIKRY'S THM

Group-free version

THM (Hrušák, Wohofsky, Zindulka 2016)

Let
- T be a σ-compact metric space
- X be a seperable locally cpt metric space
- $\phi : T \times X \to X$ continuous

  $\forall y \in T \dots \phi(\{y\} \times P)$ nowhere dense whenever $P \subseteq X$ is nowhere dense

If $S \subseteq T$ is Smz and $M \subseteq X$ is meager, then $\phi(S \times M) \neq X$.

$\tau = \mathcal{G}$

COR 2

Let $\phi: \mathcal{G} \curvearrowright X$ be an action of a σ-compact Polish group on a σ-compact space X. If $S \subseteq \mathcal{G}$ is Smz and $M \subseteq X$ meager, then $\phi(S \times M) \neq X$.

$X = T = \mathcal{G}$, $\phi(x,y) = xy$ (or $x^{-1}y$, $yx \dots$)

COR 3

(Kysiak 2000, Fremlin 2008)

Let $\mathcal{G}$ be a σ-compact Polish group. If $S \subseteq \mathcal{G}$ is Smz and $M \subseteq \mathcal{G}$ meager, then

- $SM \neq \mathcal{G}$
- $MS \neq \mathcal{G}$
- S^{-1} is Smz ...

LEMMA (Simplified version)

Let
- K, X be compact metric spaces
- $P \subseteq X$ compact nwd
- $\phi : K \times X \to X$ ... $\forall y \in K \quad \phi(\{y\} \times P)$ nwd

Then

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \; \forall y \in K \; \exists z \in X \quad B(z, \delta) \subseteq B(z, \varepsilon) \setminus \phi(B(y, \delta) \times P)$$

Proof ?

$\Rightarrow \forall \varepsilon \; \forall x \; \forall y \; \exists \delta' \; \exists z \quad B(z, \delta') \subseteq B(z, \varepsilon) \setminus \phi(\{y\} \times P)$

$\Rightarrow \forall \varepsilon \; \forall x \; \forall y \; \exists \delta \; \exists z \quad B(z, \delta) \subseteq B(z, \varepsilon) \setminus \phi(B(y, \delta) \times P)$

$\Rightarrow$ compactness:

switch quantifiers

$B(z, \delta) \subseteq B(z, \varepsilon) \setminus \phi(B(y, \delta) \times P)$

$B(z, \delta)$ ↓ deflate

$B(y, \delta)$ ↑ inflate

PROOF OF THE THEOREM

Simplified version:

- Υ_1 X compact, $\phi : \Upsilon \times X \to X$
- $P \subseteq X$ nwd, $S \subseteq \Upsilon$ Smz
- want: $\phi(S \times P) \neq X$

- recursion: $\langle \varepsilon_n \rangle$ such that

$$\forall x \; \forall y \quad \exists z \qquad B(z, \varepsilon_{n+1}) \subseteq B(x, \varepsilon_n) \setminus \phi(B(y, \varepsilon_{n+1}) \times P)$$

- S is Smz: $\exists \{E_n\}$ $\langle \varepsilon_n \rangle$-fin cover of S

$$\hookrightarrow \subseteq B(x_n, \varepsilon_n)$$

$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n) \setminus \phi(E_{n+1} \times P)$$

- $\bigcap_n B(x_n, \varepsilon_n) = \{z\}$ ...

$$\ldots \; z \notin \bigcup_n \phi(E_n \times P) \supseteq \phi(S \times P)$$

UNIFORMLY MEAGER SETS

DEF (HWZ 2016)

→ $A \subseteq X$ is uniformly nowhere dense if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \exists y \in X \ \ B(y, \delta) \subseteq B(x, \varepsilon) \setminus A$$

→ $A \subseteq X$ is uniformly meager if it is a countable union of uniformly nwd sets

→ A σ-compact meager set is uniformly meager

→ The proof works for unif. meager

THM (HWZ 2016) Let G be a TSI Polish group.

If $S \subseteq G$ is Smz, then

$$SM \neq G \quad \text{for each uniformly meager } M$$

- If X is locally compact then
  meager = uniformly meager
- If X is nowhere locally compact then
  meager $\neq$ uniformly meager
- Polish group is locally compact $\Leftrightarrow$
  meager = uniformly meager

Prop (HWZ2016)

(CH) There is $X \subseteq \mathbb{Z}^\omega$ such that

$\rightarrow$ $X + M \neq \mathbb{Z}^\omega$ for all unif. meager M

$\rightarrow$ X is not Sm2

NON - LOCALLY COMPACT CASE

- There are several proofs:
  - recursion over ω
  - recursion over $\omega^{<\omega}$
  - from the Galvin's game

All of them require compactness

- meager $\Rightarrow$ uniformly meager fails
- Borel Conjecture: GMS Thm holds for all groups trivially

- What about CH ?

ANTI - GMS SET

~~DEF~~ (HWZ 2016)

A nonempty, closed set $M \subseteq G$ is antiGMS if

→ M is nowhere dense

→ $\forall \langle \varepsilon_n \rangle \ \exists \{E_n\}$ $\langle \varepsilon_n \rangle$ - fine such that

$\forall z \in G \quad z \cdot (\bigcup_n E_n) \cap M$ is dense in M

THM (HWZ 2016)

(CH) If G contains an antiGMS set M,

then $\exists S \subseteq G$ Smz such that $SM^{-1} = G$.

→ So GMS Theorem fails for G.

Proof

- $\mathbb{G} = \{x_\alpha : \alpha < \omega_1\}$ enumerates $\mathbb{G}$
- $\{\langle \varepsilon_n^\alpha \rangle : \alpha < \omega_1\}$ enumerates $(0,\infty)^\omega$
- $\alpha < \omega_1 \mapsto \langle z_n^\alpha : n \in \omega \rangle$ $\forall z$ $z \cdot \bigcup_n B(z_n^\alpha, \varepsilon_n^\alpha) \cap M$ comeager in M

$$U_\alpha = \bigcup_n B(z_n^\alpha, \varepsilon_n^\alpha)$$

- Baire Category Thm: $\forall \alpha \; \exists m_\alpha \in M \cap x_\alpha^{-1} \cdot \bigcap_{\beta \le \alpha} U_\beta$

  → $y_\alpha := x_\alpha m_\alpha \in \bigcap_{\beta \le \alpha} U_\beta$

- $S := \{y_\alpha : \alpha < \omega_1\}$
  - → $x_\alpha = y_\alpha m_\alpha$, hence $SM^{-1} = \mathbb{G}$
  - → Let $\langle \varepsilon_n \rangle \in (0,\infty)^\omega$. $\langle \varepsilon_{2n} \rangle = \langle \varepsilon_n^\alpha \rangle$ ⟶ listed
    - → $\forall \beta \ge \alpha$ $y_\beta \in U_\alpha$
    - → $X \setminus U_\alpha \subseteq \{y_\beta : \beta < \alpha\}$ countable
  - → $S \subseteq \underbrace{\{y_\beta : \beta < \alpha\}}_{\subseteq \bigcup_n B(t_n, \varepsilon_{2n+1})} \cup \underbrace{U_\alpha}_{\subseteq \bigcup_n B(x_n^\alpha, \varepsilon_{2n})}$

THM (HWZ 2016)

$\mathbb{Z}^\omega$ contains an antiGMS set.

→ enumerate $\mathbb{Z}^{<\omega} = \{s_n : n \in \omega\}$

→ $\langle \sigma_n : n \in \omega \rangle$ such that

→ $s_n \subseteq \sigma_n$

→ $\forall m$ at most one $\sigma_n \in \mathbb{Z}^m$

→ $G = \bigcup_n \langle \sigma_n \rangle$ is open, dense but sparse

→ $M = \mathbb{Z}^\omega \setminus G$ closed nwd

→ M is antiGMS

CoRo (CH) GMS Theorem fails for $\mathbb{Z}^\omega$

SUBMEASURES

↳ Polish group

DEF $\mu : \mathcal{P}(G) \longrightarrow [0,\infty]$ is a **submeasure** if it is

- σ-subadditive
- outer-regular: $\mu(A) = \inf\{\mu(U) : A \subseteq U \text{ open}\}$
- non-atomic: $\forall x \in G \quad \mu(\{x\}) = 0$
- left-invariant: $\mu(xA) = \mu(A)$

⟶ If G is σ-compact: Haar measure is a submeasure

THM (Hrušák, Zapletal 2016)
There is a nontrivial submeasure on every Polish group.

↳ Polish group

THM (Grebík) Let $S \subseteq G$. TFAE

(i) S is Smz

(ii) $\mu(S) = 0$ for each submeasure on G.

THM (Hrušák, Zapletal 2016)

Every TSI, not locally compact Polish group contains an antiGMS set.

COR (CH) If G is a TSI Polish group, then TFAE

(i) GMS Theorem holds for G

(ii) G is locally compact

THM (Hrušák, Zapleta 2016)

Every non-locally-compact closed subgroup of S_∞ contains an antiGMS set.

Questions:

→ Does the dichotomy extend beyond TSI ?

→ What should replace "meager" to get GSM Theorem for all Polish groups ?